Quantum Chaos and Black Holes: the Lyapunov Exponent and Butterfly Velocity

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BRIEF REVIEW OF QUANTUM CHAOS

While classical chaos is relatively well understood, the characterization of quantum chaos is fairly complicated. Possible approaches include, for instance, the eigenstate thermalization hypothesis (ETH) [1], level spacing statistics described by the random matrix theory (RMT) [2], and out-of-time order correlators (OTOCs) [3]. Despite the insights provided by these approaches, a more satisfactory understanding of quantum chaos remains elusive. Here we briefly review the concepts of ETH and level spacing statistics.

Eigenstate Thermalization Hypothesis

The ETH seeks to explain how some isolated quantum systems thermalize, i.e., how an initially far-from-equilibrium state evolves to a state that appears to be in thermal equilibrium.

ETH proposes that the energy eigenstates of chaotic many-body quantum systems are indistinguishable from a thermal state when probed by local operators [1].

Level Spacing Statistics

Another feature of quantum chaotic systems is level spacing statistics described by RMT [2]. This basically states that the (properly normalized) spacing between consecutive energy eigenvalues obeys a Wigner-Dyson distribution, which also describes the level spacing statistics of random matrices. By contrast, in integrable systems, the level spacing statistics is usually described by a Poisson distribution, but there are exceptions.

OTOCS AS A DIAGNOSIS OF QUANTUM CHAOS

General definition

First, let us explain OTOCs in quantum mechanics. We would like to measure the chaotic effect between two operators $W(t)$ and $V(0)$ by computing a thermal expectation value of the square of a commutator. This expectation value can be expanded into four pieces:

$$\begin{aligned} -\langle [W(t),V(0)]^2\rangle = \langle W(t)V(0)V(0)W(t)\rangle + \langle V(0)W(t)W(t)V(0)\rangle \\ -\langle W(t)V(0)W(t)V(0)\rangle - \langle V(0)W(t)V(0)W(t)\rangle. \end{aligned} \tag{1}$$

The last two terms are four-point OTOCs in quantum mechanics.

Next, consider OTOCs $\langle W(t,x)V(0)W(t,x)V(0)\rangle$ in quantum field theories (QFTs), where t and x are time and space coordinates of $W(t,x)$. OTOCs in QFTs can be computed by performing an analytic continuation of Euclidean correlation functions. In this analytic continuation, the order of Euclidean time in the original Euclidean correlation functions is important for out of time order in real time.

In some chaotic systems, the OTOCs at a certain time t behave as

$$\langle W(t,x)V(0)W(t,x)V(0)\rangle \sim c_0 + c_1 \exp[\lambda(t-x/v)] \tag{2}$$

with positive λ. Now, λ is the quantum version of the Lyapunov exponent, which characterizes the divergence of initially nearby trajectories in classical chaotic motion; and v is the butterfly velocity, which describes the speed at which information propagates through space.

QFT examples

We give examples of QFTs where we can compute OTOCs. In the below examples, the Lyapunov exponent saturates the upper bound [4], and it is a characteristic property of QFTs that have the gravity duals.

The first example is the SYK model [5]. The SYK model is a quantum mechanical system with N Majorana fermions. These fermions interact with random couplings from a Gaussian distribution. In the large N and strong coupling limit, we obtain the upper bound value of the Lyapunov exponent.

The second example is seen in two-dimensional holographic conformal field theories [6]. The computation of OTOCs from the Virasoro identity block with a large central charge reproduces the holographic computation of OTOCs.

Holographic calculation of OTOCs

In the context of the gauge-gravity duality [3], OTOCs are related to a high energy shock wave collision that takes place close to a black hole horizon. The collision is dominated by the gravitational interaction, which leads to the universal chaotic behavior of the boundary theory OTOCs with $\lambda = 2\pi T$, where T is the Hawking temperature.

Holographic calculation shows that black holes saturate the chaos bound, i.e., $\lambda \leq 2\pi T$. This singles out holographic systems as being maximally chaotic. This special property of holographic systems helped in the construction of simple models for holography, e.g., SYK model / JT gravity [5].

The Lyapunov exponent λ and the butterfly velocity (v) fully characterize the universal part of the OTOC and are important parameters characterizing the chaotic behavior.

Chaos bound in rotating black holes

A large class of black holes is characterized by a maximal Lyapunov exponent, namely, $\lambda = 2\pi T$ [4]. In fact, it is actually very difficult to find examples of black holes that go beyond the maximal exponent; the only known example of such a black hole in which the Lyapunov exponent does not take such a simple form is provided by the rotating BTZ black holes. In this case, the angular momentum affects the Lyapunov exponent, leading to contributions that either fail to saturate the chaos bound, or violate it [7].

POLE-SKIPPING PHENOMENON

The parameters λ and v can also be extracted from momentum-space energy-density retarded two-point functions $G(\omega,k)$. Here, ω is the frequency and k the spatial momentum.

By writing $G(\omega,k) = b(\omega,k)/a(\omega,k)$, one can find the poles and the zeros of $G(\omega,k)$ as the zeros of $a(\omega,k)$ and $b(\omega,k)$ respectively. There are, however, some special points (ω^*,k^*) at which $a(\omega^*,k^*) = b(\omega^*,k^*)=0$. Those are called pole-skipping points, and they were shown to be related to λ and v [8]:

$$\omega^* = i\,2\,\pi T, \qquad k^* = i\,2\pi\,T/v \tag{3}$$

Since $G(\omega,k)$ characterizes the transport properties of the theory, the pole-skipping phenomenon points to an interesting connection between chaos and hydrodynamics.

Pole-skipping in hyperbolic black holes

The pole-skipping phenomenon was first observed/studied for planar black holes, but it was later realized that the pole-skipping phenomenon also happens for hyperbolic black holes [9]. In that case, the momentum-space retarded energy-density two-point function $G(\omega,L)$ is characterized by the frequency ω and by an angular-momentum-like quantum number L.

The pole-skipping points (ω^*,k^*) of $G(\omega,L)$ were shown to be related to the Lyapunov exponent and butterfly velocity extracted from OTOCs.

Pole-skipping for scalar and vector fields

The pole-skipping phenomenon also occurs for other fields rather than the gravitational sound mode that characterizes fluctuations in energy density. In fact, it was shown that it also occurs for scalar and vector fields, and also for other sectors of gravitational perturbations [10].

In [10], the authors considered a (d+1)-dimensional Rindler-AdS geometry and studied the pole-skipping points of scalar and vector fields using several different methods, including both CFT and holographic calculations. They observed a precise matching between the pole-skipping point obtained in both sides of the gauge-gravity duality. Moreover, they showed that pole-skipping points are related to the late-time behavior of conformal blocks and shadow conformal blocks.

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